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First-order necessary optimality conditions in fuzzy nonlinear programming problems

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1. Fuzzy numbers

Definition 1.1. A *fuzzy number* A is a fuzzy set with a membership function $\mu : \mathbf{R} \to [0, 1]$ satisfying the following conditions :

(i) there exists a unique real number m such that $\mu(m) = 1$,
(ii) $\mu(t)$ is upper semi-continuous on $\mathbf{R}$,
(iii) the support supp(A) of A is not a singleton, and $\mu(t)$ is strictly increasing on $(-\infty, m] \cap$ supp(A), and strictly decreasing on $[m, \infty) \cap$ supp(A),
(iv) in the case where the support of A is not bounded, it holds that

$$\lim_{t \to \pm\infty} \mu(t) = 0.$$

The set of all fuzzy numbers is denoted by $\mathcal{F}(\mathbf{R})$. Especially, the set of all fuzzy numbers whose supports are compact is denoted by $\mathcal{F}^{\mathrm{C}}(\mathbf{R})$.

We define a shape function by the following.

Definition 1.2. Let L be a function from $\mathbf{R}$ to $[0, 1]$ satisfying the following conditions :

(i) $L(t) = L(-t) \quad \forall t \in \mathbf{R}$,
(ii) $L(t) = 1$ iff $t = 0$,
(iii) $L(\cdot)$ is upper semi-continuous on $\mathbf{R}$.
(iv) supp(L) is not a singleton, and $L(\cdot)$ is strictly decreasing on $[0, +\infty) \cap$ supp(L),
(v) $\lim_{t \to +\infty} L(t) \leqq 0$.

Then the functon L is called a *shape function*.

Let L be a shape function. Let m be an arbitrary real number, and let β an arbitrary positive number. Then an L fuzzy number μ_L is defined by

$$\mu_L(t) = L(\,(t-m)/\beta\,), \qquad t \in \mathbf{R}. \tag{1.1}$$

We call m the *center* of the L fuzzy number, and call β the *deviation* of L .In place of (1.1) we will use a parametric representation, that is,

$$(m, \beta)_L.$$

Given a shape function L, the set of all L fuzzy numbers is denoted by $\mathcal{F}(\mathbf{R})_L$.

According to usual notation, for an arbitrary fuzzy number A, we denote the α-cut $\{t \in \mathbf{R} \mid \mu_A(t) \geqq \alpha\}$ of A by A_α. In the case where the support is bounded, we define the 0-cut of A by the closure of the support. We identify the shape function L with an L fuzzy number having its center 0 and its deviation 1, that is, $L = (0, 1)_L$. For each $\alpha \in [0, 1]$, we denote the right end-point of L_α by t_α^L. In the case where the support of L is not bounded, we interpret as $t_0^L = +\infty$.

Proposition 1.1. (i) For every shape function L, t_α^L is continuous with respect to α on $(0, 1]$, and monotonically decreases in the wide sense as α increases on $(0, 1]$. Especially when L has a compact support, all of these statements hold on the closed interval $[0, 1]$ not on the interval $(0, 1]$.

(ii) If L is continuous on $\mathbf{R}^n$ and has a compact support, then t_α^L is strictly decreasing as α increases on $[0, 1]$.

The following definition of an order relation on $\mathcal{F}(\mathbf{R})$ is well known and is called the *fuzzy max order*.

Definition 1.3. Let A and B be two members of $\mathcal{F}(\mathbf{R})$. Then, the relation $A \preceq B$ is defined by

$$\left(\sup A_\alpha \leqq \sup B_\alpha\right) \;\&\; \left(\inf A_\alpha \leqq \inf B_\alpha\right) \quad \text{for each } \alpha \in [0, 1]. \tag{1.2}$$

. In this paper we shall use two types of strengthened versions of the fuzzy max order as follows.

Definition 1.4. Let A and B be two members of $\mathcal{F}^c(\mathbf{R}) \cup \mathbf{R}$. Suppose that at least one of A and B belongs to $\mathcal{F}^c(\mathbf{R})$ not to $\mathbf{R}$. Then the order relation $A \prec B$ is defined by

$$\begin{cases} \left(\sup A_0 \leqq \sup B_0\right) \;\&\; \left(\inf A_0 \leqq \inf B_0\right), \\ \qquad \text{and} \\ \left(\sup A_\alpha < \sup B_\alpha\right) \;\&\; \left(\inf A_\alpha < \inf B_\alpha\right) \quad \text{for } \forall \alpha \in (0, 1], \end{cases} \tag{1.3}$$

and the order relation $A \prec\!\prec B$ is defined by

$$\left(\sup A_\alpha < \sup B_\alpha\right) \;\&\; \left(\inf A_\alpha < \inf B_\alpha\right) \quad \text{for } \forall \alpha \in [0, 1]. \tag{1.4}$$

Especially when B is a real number in the definition 1.4, we have the following proposition.

Proposition 1.2. For a fuzzy number $A \in \mathcal{F}^c(\mathbf{R})$ and a real number t, it holds that

$$A \prec t \Leftrightarrow \sup A_0 \leqq t. \tag{1.5}$$

and

$$A \prec\!\prec t \Leftrightarrow \sup A_0 < t. \tag{1.6}$$

2. One-sided directional derivatives of fuzzy mappings.

Let X be a real normed linear space. Throughout this section, U denotes an open subset of X, and Ω denotes an open convex subset of X.

Let F be a fuzzy mapping from U to $\mathcal{F}(\mathbf{R})$. Let $z \in U$ and $h \in X$. For each $\alpha \in (0,1]$, we put

$$\eta(\alpha) = \lim_{\lambda \downarrow 0} \frac{\inf F(z+\lambda h)_\alpha - \inf F(z)_\alpha}{\lambda}, \tag{2.1}$$

and

$$\xi(\alpha) = \lim_{\lambda \downarrow 0} \frac{\sup F(z+\lambda h)_\alpha - \sup F(z)_\alpha}{\lambda}, \tag{2.2}$$

supposing that these two limits exist as finite values.

For $\alpha = 0$, we put

$$\eta(0) = \lim_{\alpha \downarrow 0} \eta(\alpha), \tag{2.3}$$

$$\xi(0) = \lim_{\alpha \downarrow 0} \xi(\alpha). \tag{2.4}$$

By the definition of $\mathcal{F}(\mathbf{R})$, when $\alpha = 1$, each one of $F(z+\lambda h)_1$ and $F(z)_1$ consists of a singleton. Accordingly it holds that $\eta(1) = \xi(1)$.

We put

$$\left.\begin{aligned} i(\alpha) &= \min\big(\eta(\alpha),\ \xi(\alpha)\big), \\ s(\alpha) &= \max\big(\eta(\alpha),\ \xi(\alpha)\big). \end{aligned}\right\} \quad \alpha \in [0,1]. \tag{2.5}$$

Assumption I. The functions $\eta(\cdot)$ and $\xi(\cdot)$ are continuous on $(0,1]$.
Assumption II. $i(\cdot)$ is nondecreasing and $s(\cdot)$ is nonincreasing on $(0,1]$.

Under Assumptions I and II, define $f : \mathbf{R} \to [0,1]$ by

$$f(t) = \begin{cases} \max\{\alpha \in [0,1] \mid i(\alpha) = t\} & \text{if } i(0) < t \leqq i(1)\ (= s(1)), \\ \max\{\alpha \in [0,1] \mid s(\alpha) = t\} & \text{if } s(1) \leqq t < s(0), \\ 0 & \text{if otherwise,} \end{cases} \tag{2.6}$$

for $t \in \mathbf{R}$.

When Assumptions I and II are satisfied, it is easily verified that f defined by (2.6) is qualified as a membership function.

Definition 2.1. When Assumptions I and II are satisfied, the fuzzy number f given by (2.6) is called *the one-sided directional derivative* of F *at* z *in the direction* h, and is denoted by $F'(z\,;h)$, and then F is said to be *one-sided directionally differentiable at* z *in the direction* h. If F is one-sided directioally differentiable at z in every direction h, then F is said to be *one-sided directionally differentiable at* z.

Definition 2.2. ([3]) Let F be a mapping from U to $\mathcal{F}^c(\mathbf{R})$. Let x_0 be a point of U. Then F is said to be *continuous* at x_0 iff for each $\varepsilon > 0$ there exists a neigh- borhood $U(x_0)$ of x_0 satisfying that

$$F(x_0)-\varepsilon \preceq F(x) \preceq F(x_0)+\varepsilon \quad \forall x \in U(x_0).$$

Definition 2.3. ([3]) A mapping F from Ω to $\mathcal{F}^{c}(\mathbf{R})$ is said to be *convex* on Ω iff for every $x, y \in \Omega$ and every $\lambda \in (0,1)$ it holds that

$$F(\lambda x+(1-\lambda)y) \preceq (\lambda \cdot F(x)) \oplus ((1-\lambda)\cdot F(y)), \tag{2.7}$$

where $\oplus$ and $\cdot$ are the addition and the multiplication, respectively, defined by the usual extension principle. For the sake of simplicity we write (2.7) as

$$F(\lambda x+(1-\lambda)y) \preceq \lambda F(x) \oplus (1-\lambda) F(y). \tag{2.8}$$

Proposition 2.1. Let $F: \Omega \to \mathcal{F}^{c}(\mathbf{R})$ be a convex mapping. Then, for every $z \in U$, $h \in X$ and for each $\alpha \in [0,1]$, the limits $\eta(\alpha)$ and $\xi(\alpha)$ defined by (2.1) and (2.2), respectively, exist as finite values.

Theorem 2.1. Let L be a shape function whose support is compact. Let F be a convex mapping from Ω to $\mathcal{F}(\mathbf{R})_L$, and let the parametric representation of the mapping be given by

$$\left.\begin{aligned} F(x) &= \big(m(x), \beta(x)\big)_L, \\ \beta(x) &\geqq 0, \end{aligned}\right\} \quad x \in \Omega.$$

Then we have

(i) both $m(\cdot)$ and $\beta(\cdot)$ are one-sided directionally differentiable in the usual sense at all $z \in U$ and in every direction $h \in X$, and for each $\alpha \in [0,1]$, $\eta(\alpha)$ and $\xi(\alpha)$ can be expressed as

$$\eta(\alpha) = m'(z\,;h) - \beta'(z\,;h)t_\alpha^L, \tag{2.9}$$

$$\xi(\alpha) = m'(z\,;h) + \beta'(z\,;h)t_\alpha^L, \tag{2.10}$$

where $m'(z\,;h)$ and $\beta'(z\,;h)$ denote the one-sided directional derivatives (in the usual sense) of m and β, respectively,

(ii) for every $z \in U$ and every $h \in X$, F is one-sided directionally differentiable at z in the direction h, and the directional derivative of F is expressed as

$$F'(z\,;h) = \big(\, m'(z\,;h),\, |\beta'(z\,;h)|\,\big)_L. \tag{2.11}$$

Theorem 2.2. Let L be an arbitrary shape function. Let F be a mapping from an open subset U of $\mathbf{R}^n$ to $\mathcal{F}(\mathbf{R})_L$, having its parametric representation

$$F(x) = \big(m(x), \beta(x)\big)_L, \qquad x \in U,$$

$$\beta(x) \geqq 0, \qquad x \in U.$$

Suppose that $m(\cdot)$ and $\beta(\cdot)$ are differentiable in the usual meaning on U. For every $z \in U$ and every $h \in X$, then, F is one-sided directionally differentiable at z in the direction h, and the directional derivative of F is expressed as

$$F'(z\,;h) = \big(\, \nabla m(z)h,\, |\nabla \beta(z)h|\,\big)_L. \tag{2.12}$$

3. Fuzzy nonlinear programming.

3.1. The unconstrained problem.

Let F be a mapping from $\mathbf{R}^n$ to $\mathcal{F}(\mathbf{R})$. We consider the following unconstrained minimization problem:

$$\text{(P1)} \qquad \underset{x \in \mathrm{R}^n}{\text{Minimize}}\; F(x),$$

where the minimization is taken in the meaning of the fuzzy max order.

Definition 3.1. A point $z \in \mathbf{R}^n$ is called a *local minimum solution* to (P1), if there exists a neighborhood V of z such that

$$F(z) \preceq F(x) \quad \forall x \in V. \tag{3.1}$$

A point $z \in \mathbf{R}^n$ is called a *global minimum solution* to (P1), if (3.1) holds for all $x \in \mathbf{R}^n$.

Theorem 3.1. Let L be an arbitrary shape function. Let F be a mappimg from $\mathbf{R}^n$ to $\mathcal{F}(\mathbf{R})_L$ with the parametric representation:

$$\left.\begin{aligned} F(x) &= \big(m(x), \beta(x)\big)_L, \\ \beta(x) &\geqq 0, \end{aligned}\right\} \quad x \in \mathbf{R}^n.$$

Suppose that $m(\cdot)$ and $\beta(\cdot)$ are differentiable on $\mathbf{R}^n$. If z is a local minimum solution to (P1), then it holds that

$$\begin{cases} \nabla m(z) = \mathbf{0}, \\ \nabla \beta(z) = \mathbf{0}. \end{cases} \tag{3.2}$$

3.2. The problem with inequality constraints.

Let F be a mapping from $\mathbf{R}^n$ to $\mathcal{F}^{\mathrm{c}}(\mathbf{R})$, and let $G_1, G_2, \cdots, G_m$ be m mappings from $\mathbf{R}^n$ to $\mathcal{F}^{\mathrm{c}}(\mathbf{R})$. Let $B_1, B_2, \cdots, B_m$ be m elements of $\mathcal{F}^{\mathrm{c}}(\mathbf{R})$. Then we consider the following problem.

$$\text{(P2)} \qquad \begin{cases} \text{Minimize } F(x) \\ \quad \text{subject to} \\ G_i(x) \preceq B_i, \quad i = 1, 2, \cdots, m. \end{cases}$$

Define the set of all feasible solutions to (P2) by

$$S = \{\, x \in \mathbf{R}^n \mid G_i(x) \preceq B_i, \quad i = 1, 2, \cdots, m \,\}.$$

Definition 3.2. A point $z \in S$ is called a *local minimum solution* to (P2), if there exists a neighborhood V of z such that

$$F(z) \preceq F(x) \quad \forall x \in V \cap S. \tag{3.3}$$

A point $z \in S$ is called a *global minimum solution* to (P1), if (3.3) holds for all $x \in S$.

Definition 3.3. A point $z \in S$ is called a *local nondominated solution* to (P2), if there exists a neighborhood V of z such that there is no point x in $V \cap S$ satisfying both of $F(x) \preceq F(z)$ and $F(x) \neq F(z)$.

Definition 3.4. A point $z \in S$ is called a *local weak nondominated solution* to (P2), if there exists a neighborhood V of z such that there is no point x in $V \cap S$ satisfying $F(x) \prec F(z)$.

Proposition 3.1. (i) If $z \in S$ is a local minimum solution to (P2), then z is a local nondominated solution.

(ii) If $z \in S$ is a local nondominated solution to (P2), then z is a local weak nondominated solution.

Proposition 3.2. Let L be a shape function which is continuous on its compact support. Let F be a mappimg from $\mathbf{R}^n$ to $\mathcal{F}(\mathbf{R})_L$. Then, for a point $z \in S$, the statements (i) and (ii) are equivalent to each other:

(i) z is a local nondominated solution to (P2).

(ii) z is a local weak nondominated solution to (P2).

Proposition 3.3. Let L be same as in Proposition 3.2. Let A and B be two members of $\mathcal{F}(\mathbf{R})_L$ such that $A \preceq B$. Then, for the pair A and B, the statements (i), (ii) and (iii) are equivalent one another:

(i) There exists a number $\alpha_0 \in [0, 1]$ such that

$$\begin{cases} \inf A_{\alpha_0} = \inf B_{\alpha_0} \\ \quad \text{or} \\ \sup A_{\alpha_0} = \sup B_{\alpha_0}. \end{cases}$$

(ii) Either one and only one of the following three statements holds:

(ii - 1) $A = B$,

(ii - 2) $(\inf A_\alpha < \inf B_\alpha) \,\&\, (\sup A_\alpha < \sup B_\alpha)$ for all α except for $(\inf A_0 = \inf B_0)$,

(ii - 3) $(\inf A_\alpha < \inf B_\alpha) \,\&\, (\sup A_\alpha < \sup B_\alpha)$ for all α except for $(\sup A_0 = \sup B_0)$.

(iii) Either one of the following three statements holds:

(iii - 1) $\mathrm{A}_1 = B_1$,

(iii - 2) $\inf A_0 = \inf B_0$,

(iii - 3) $\sup A_0 = \sup B_0$.

In the remainder of this section we assume the following assumptions to the problem (P2). For each $i = 0, 1, \cdots, m$, let L_i be a shape function which has a compact support and is continuous on its support. We asssume that F is a mapping from $\mathbf{R}^n$ to $\mathcal{F}(\mathbf{R})_{L_0}$, and that, for each $i = 1, 2, \cdots, m$, G_i is a mapping from $\mathbf{R}^n$ to $\mathcal{F}(\mathbf{R})_{L_i}$, and B_i is an element of $\mathcal{F}(\mathbf{R})_{L_i}$. Let

$$\left.\begin{aligned} F(x) &= \left(m_0(x), \beta_0(x)\right)_{L_0}, \\ \beta_0(x) &\geqq 0, \end{aligned}\right\} \quad x \in \mathbf{R}^n, \tag{3.4}$$

$$\left.\begin{aligned} G_i(x) &= \left(m_i(x), \beta_i(x)\right)_{L_i}, \\ \beta_i(x) &\geqq 0, \end{aligned}\right\} \quad x \in \mathbf{R}^n, \quad i = 1, 2, \cdots, m. \tag{3.5}$$

Now, for an arbitrary feasible solution $x \in S$, we define the two kinds of index sets as follows :

$$\tilde{I}(x) = \left\{ i \in \{1,2,\cdots,m\} \;\middle|\; \exists \alpha_i \in [0,1]\,; \left\{\begin{array}{c} \inf G_i(x)_{\alpha_i} = \inf (B_i)_{\alpha_i} \\ \text{or} \\ \sup G_i(x)_{\alpha_i} = \sup (B_i)_{\alpha_i} \end{array}\right\} \right\},$$

and

$$I(x) = \left\{ i \in \{1,2,\cdots,m\} \;\middle|\; \begin{array}{l} \text{(a)}\;\; G_i(x)_1 = (B_i)_1, \\ \qquad \text{or} \\ \text{(b)}\;\; \inf G_i(x)_0 = \inf (B_i)_0, \\ \qquad \text{or} \\ \text{(c)}\;\; \sup G_i(x)_0 = \sup (B_i)_0 \\ \quad \text{holds true.} \end{array} \right\}.$$

Then, it is easily seen from Proposition 3.3 that the relation

$$\tilde{I}(x) = I(x) \tag{3.6}$$

holds. We call $I(x)$ *the index set for the binding constraints at* x in the problem (P2). Owing to (3.6), $\tilde{I}(x)$ is also qualified to be called the binding set. But, $I(x)$ is more suitable than $\tilde{I}(x)$ in order to descript optimality conditions as seen later on. The index set $\tilde{I}(x)$ is used only to develop our arguments in the course of deriving optimality conditions.

The following proposition gives a first-order necessary optimality condition in a primal form.

Proposition 3.4. Let $z \in S$ be a local nondominated solution to (P2). Suppose that the following two assumptions hold :

(i) In (3.4) and (3.5), $\{m_i(\cdot)\}$ and $\{\beta_i(\cdot)\}$ are all differentiable on $\mathbf{R}^n$.

(ii) For each $i \notin I(z)$, G_i is continuous at z.

Define

$$\Psi(z)=\left\{h\in\mathbf{R}^n \mid F'(z\,;h)\prec\!\prec 0\right\}$$

and

$$\Gamma(z)=\left\{h\in\mathbf{R}^n \mid G_i'(z\,;h)\prec 0 \quad \forall i\in I(z)\right\}.$$

Then it holds that

$$\Psi(z)\cap\Gamma(z)=\phi\,. \tag{3.7}$$

When $\Gamma(z)=\phi$, the necessary optimality conditon (3.7) holds vacuously. For this reason, we set up a constraint qualification as follows.

Constraint Qualification at z : $\Gamma(z)\neq\phi$.

Under the assumption of Constraint Qualification, first-order necessary optimality conditions in a dual form are given by the following.

Theorem 3.2. Let $z\in S$ be a local nondominated solution to (P2). Suppose the assumptions (i) and (ii) in Proposition 3.4 to be satisfied. Under Constraint Qualification at z, then, there exist multipliers $\{\mu_i\,;i=0,1,\cdots,m\}$ and $\{\lambda_i\,;i=1,2,\cdots,m\}$ satisfying that

$$\nabla m_0(z)+\mu_0 t_0^{L_0}\nabla\beta_0(z)+\sum_{i=1}^{m}\lambda_i\nabla m_i(z)+\sum_{i=1}^{m}\mu_i t_0^{L_i}\nabla\beta_i(z)=\mathbf{0}, \tag{3.8}$$

$$\lambda_i\geqq 0,\ \ i=1,2,\cdots,m, \tag{3.9}$$

$$\begin{aligned}&\lambda_i\left(m_i(z)-(B_i)_1\right)\left(m_i(z)-t_0^{L_i}\beta_i(z)-\inf(B_i)_0\right)\\ &\qquad\times\left(m_i(z)+t_0^{L_i}\beta_i(z)-\sup(B_i)_0\right)=0,\quad i=1,2,\cdots,m.\end{aligned} \tag{3.10}$$

$$\begin{aligned}&\mu_i\left(m_i(z)-(B_i)_1\right)\left(m_i(z)-t_0^{L_i}\beta_i(z)-\inf(B_i)_0\right)\\ &\qquad\times\left(m_i(z)+t_0^{L_i}\beta_i(z)-\sup(B_i)_0\right)=0,\quad i=1,2,\cdots,m.\end{aligned} \tag{3.11}$$

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